Martingale integrals over Poissonian processes and the Ito-type equations with white shot noise

Ryszard Zygadło
Marian Smoluchowski Institute of Physics, Jagiellonian University, Reymonta 4, PL–30059 Kraków, Poland
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The construction of the Ito-type stochastic integrals and differential equations for compound Poisson processes is provided. The general martingale and nonanticipating properties of the ordinary (Gaussian) Ito theory are conserved. These properties appear particularly important if the stochastic description has to be proposed according to game theory or the linear relaxation (or the exponential growth) requirements. In contrast to the ordinary Ito theory the (uncorrelated) parametric fluctuation of a definite sign can be still modeled by asymmetric white shot noise, so the general scope of applications is not restricted by the positivity requirements. The possible use of the developed formalism in econophysics is addressed.

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I. INTRODUCTION

The Ito theory of stochastic integrals (SI) and stochastic differential equations (SDE) [1,2] results with *martingale* properties of the integrals over the Wiener process and related *nonanticipating* properties of stochastic (Langevin) equations with (multiplicative) Gaussian white noise (GWN), which appear useful in a number of applications and model studies. In Ref. [3] the importance of simultaneous regarding of single-event properties [determined by a given (required) probability distribution] together with certain correlation behavior in stochastic modeling of various dynamical systems has been pointed out. The main idea of the construction proposed in Ref. [3] is to choose the drift and diffusion term in the Ito SDE in a way to fit both the target stationary probability density and the required autocorrelation function (or, equivalently, the spectral density) form. The method has then been used to generate stationary non-Gaussian (Markov) processes with exponentially decaying correlation (or low-pass spectral density) [4,5], considered as particularly important in the Monte Carlo studies. Such behavior of correlation appears fixed by the linear drift term of the Ito SDE, irrespectively on the form of the (state-dependent) diffusion term. The last actually follows from the general (nonanticipating) property of the Ito SDE:

$$dx_t = f(x)dt + g(x)d\circ W_t, \tag{1}$$

the $\circ$ sign is to indicate that the equation is understood (*interpreted*) according to the Ito definition that $\langle g(x)d\circ W_t\rangle = 0$, so that the regression equation for average

$$d\langle x_t\rangle = \langle f(x)\rangle dt$$

looks similar to the "deterministic" one $dx = f(x)dt$, and becomes the same for an affinic f, i.e., when $\langle f(x)\rangle = f(\langle x\rangle)$.

Ito equation (1) of a latter type, with $f(x) = rx$ and $g(x) = \sigma x$, appears as a basic equation of (Bachelier-) Black-Scholes theory of financial markets [6,7], and it is obtained under the assumptions that the average return is determined by the *interest rate* r and that the relative changes of prices are independent and Gaussian (the parameter σ is called *volatility*). The Ito interpretation warrants that the deterministic trend is not affected by the presence of fluctuations (*risk*). The *discounted* price $\tilde{S}_t$, $x_t \equiv S_t = e^{rt}\tilde{S}_t$, satisfies the (particular) *driftless* Ito equation $d\tilde{S}_t = \sigma\tilde{S}d\circ W_t$. Such processes have the property that the (future) conditional mean value, $\langle x_t | x_{t_n}, \ldots, x_{t_1}\rangle = x_{t_n}$, $t \geqslant t_n \geqslant \cdots \geqslant t_1$, is equal to the last value specified by the condition. Such processes are generally called *martingales* and they are used for game theory to define the *fairness* of the game. The Ito theory provides an effective method of construction of a certain class of martingales.

The use of the Ito theory is limited to Gaussian fluctuation. In some cases of (especially nonlinear) kinetic equations, such unbounded *parametric* fluctuations are excluded by the stability conditions or by the positivity requirements [8] and then the other noise (not the GWN) is needed to describe the fluctuation. On the other hand, the well-known property of the Ito theory is that the ordinary rules of differentiation and integration are no longer valid [1,2], being replaced by the specific Ito calculus. The mentioned general properties of the Ito theory appear precisely related to this specific calculus and thus cannot be immediately implemented to the non-Gaussian theory. The proper construction of the Ito-type theory based on Poissonian processes is the main aim of the paper.

Let us remind that the integral over the Wiener process

$$J_t = \int_0^t H(W_s)dW_s$$

is not precisely determined in a conventional (Stieltjes-type) sense, unless the additional rule of choosing the intermediate points during construction of approximate sums is specified [2]. The Ito choice

$$J_t = \lim \sum H(W_{s_i})[W_{s_{i+1}} - W_{s_i}]$$

results, due to the statistical independence of increments of the Wiener process, in martingale property of the integral $\langle J_t\rangle = 0$. As a consequence, the ordinary rules do not apply for the Ito integrals. In fact, having $\langle I_t\rangle = 0$, where

$$I_t=\int_0^t G'(W_s)d\circ W_s$$

we cannot still identify I_t with $\Delta G=G(W_t)-G(0)$, $I_t \neq \Delta G$, because in general $\langle \Delta G\rangle \neq 0$. Similarly, it turns out that $x_t \equiv x(t,W_t)$, considered as a function of two variables, represents the solution of Ito equation (1) only if the usual $\partial x/\partial W=g(x)$ and the *unusual* $\partial x/\partial t=f(x)-Dg(x)g'(x)$ condition is satisfied. [The noise strength D is normalized by Eq. (3) or (5).]

There exists a Stratonovich definition of SI [9,2], which is free of this inconvenience, i.e., which is consistent with the ordinary rule $dG(W_s)=G'(W_s)dW_s$ (in our notation without $\circ$ sign),

$$J_t=\lim\sum H(W_{(s_{i+1}+s_i)/2})[W_{s_{i+1}}-W_{s_i}].$$

This formal definition of SI corresponds to the choice of values of the integrand from the middle of the successive time intervals. Its practical importance lies in that the usual methods of computing the integrals and especially solving the differential equations remain unchanged. For instance, the ordinary calculus applies also to the SDE written in Stratonovich interpretation. It means that Ito equation (1) and the Stratonovich equation

$$dx_t=[f(x)-Dg(x)g'(x)]dt+g(x)dW_t$$

have the same solutions $x(t,W_t)$ and in such sense both forms are equivalent. The term Dgg' is called the "spurious drift." The Stratonovich definition is more popular in a physical literature because it may be easily extended for integrals over some other "singular" processes. The advantage of the Ito approach is that it usually better includes certain general probabilistic properties required from the particular stochastic description (modeling). The advantage of the Stratonovich approach is that the well recognized (ordinary) methods of transforming the variables and solving the differential equations can be used. The two approaches may be thus considered complementary.

The paper is organized as follows. In Sec. II the Ito-type integrals over compound Poisson processes are defined, in Sec. III the relation transforming between Ito-type and Stratonovich-type of differential equations is obtained. The examples are presented. The last section is for remarks.

II. STOCHASTIC INTEGRALS OVER POISSON PROCESSES

There are two fundamental stochastic processes, Wiener process W_t and Poisson process N_t. The first is Gaussian and continuous, the second is a point process and its trajectories are piecewise constant functions, having some steps in random points t_i, which are distributed on positive semiaxis with average frequency λ

$$N_t=\sum_i z_i\Theta(t-t_i). \tag{2}$$

z_i are independent random numbers with common probability distribution $\pi(z)$. Both processes have stationary and independent increments. Using standard normalization (particularly $\langle W_t^2\rangle=2Dt$, in the case of the Wiener process) we have

$$\langle \exp(yW_t)\rangle=\exp(Dty^2), \tag{3}$$

$$\langle \exp(yN_t)\rangle=\exp[\lambda t(-1+Ee^{yz})], \tag{4}$$

where $Ea(z)=\int dz\,\pi(z)a(z)$ means averaging over random weights z. Thus, for the infinitesimal increments (due to the stationarity) $dW=W_{dt}$, $dN=N_{dt}$, $k=1,2,\dots,$ and

$$\langle (dW)^k\rangle=\delta_{2,k}2Ddt+o[(dt)^2], \tag{5}$$

$$\langle (dN)^k\rangle=\lambda dtEz^k+o[(dt)^2]. \tag{6}$$

Note that $\langle N_t\rangle=\lambda tEz$, so it is often convenient to subtract the deterministic compensator and consider the new process $\tilde{N}_t=N_t-\lambda tEz$ (of a zero mean).

Consider the Stratonovich integral

$$\int_0^t dG(N_s)=G(N_t)-G(0)$$

and calculate its mean value. Because

$$\begin{aligned}\langle dG(N)\rangle&=\langle G(N+dN)-G(N)\rangle\\&=\sum_{k=1}^{\infty}\left\langle\frac{d^kG}{dN^k}\right\rangle\frac{\langle (dN)^k\rangle}{k!}\\&=\sum_{k=1}^{\infty}\left\langle\frac{d^kG}{dN^k}\right\rangle\frac{\lambda dsEz^k}{k!}\\&=\lambda dsE\langle G(N+z)-G(N)\rangle,\end{aligned} \tag{7}$$

thus

$$\langle G(N_t)-G(0)\rangle=\int_0^t ds\lambda E\langle G(N_s+z)-G(N_s)\rangle. \tag{8}$$

Computing Eq. (7) we have used stationarity and independence of increments of N_t and Eq. (6).

According to the Stratonovich idea

$$dG(N_s)=G'(N_s)dN_s, \tag{9}$$

and consequently for the Ito interpretation

$$dG(N_s)=G'(N_s)d\circ N_s+\lambda[EG(N_s+z)-G(N_s)]ds. \tag{10}$$

In conclusion, we have following integration formulas:

$$\int_0^t G'(N_s)dN_s=G(N_t)-G(0), \tag{11}$$

for the Stratonovich-type integral and

$$\int_0^t G'(N_s)d\circ N_s = G(N_t) - G(0) - \lambda\int_0^t ds[EG(N_s+z) - G(N_s)] \qquad (12)$$

for Ito-type integral.

The limiting procedure [10]

$$Ez=0, \quad z\to 0, \quad \lambda\to\infty, \quad \lambda Ez^2 = 2D, \qquad (13)$$

in which a compound Poisson process approaches the Wiener process applied to Eq. (12) recovers the Ito rule of integration

$$\int_0^t G'(W_s)d\circ W_s = G(W_t) - G(0) - D\int_0^t ds G''(W_s).$$

Also note the expression

$$H(N_s)dN_s = H(N_s)d\circ N_s + \lambda ds E\int_{N_s}^{N_s+z} du H(u), \qquad (14)$$

which results from Eqs. (9) and (10).

Examples. Let us give few examples of the Ito-type integrals over standard Poisson process with $z\equiv 1$. Let $G(u) = u$. Then, from Eq. (12)

$$\int_0^t d\circ N_s = N_t - \lambda t. \qquad (15)$$

Similarly, for $G(u) = u^2$

$$\int_0^t 2N_s d\circ N_s = N_t^2 - \lambda\int_0^t ds[(N_s+1)^2 - N_s^2] = N_t^2 - \lambda t - 2\lambda\int_0^t ds N_s. \qquad (16)$$

Using $\langle N_t^2\rangle = \lambda^2 t^2 + \lambda t$ and $\langle N_s\rangle = \lambda s$ we verify that the mean value of Eq. (16) indeed vanishes. Consider finally $G(u) = \exp(yu)$. One obtains

$$\int_0^t y e^{yN_s} d\circ N_s = e^{yN_t} - 1 - \lambda(e^y - 1)\int_0^t ds e^{yN_s}. \qquad (17)$$

Using *explicit* expressions for averages, see Eq. (4), we find again that the mean value of the right-hand side of Eq. (17) is zero.

III. STOCHASTIC DIFFERENTIAL EQUATIONS

Consider a (Stratonovich-type) equation

$$dx_t = f(x)dt + g(x)dN_t, \qquad (18)$$

where $x_t \equiv x(t,N_t)$ satisfying $\partial x/\partial t = f(x(t,N))$ and $\partial x/\partial N = g(x(t,N))$ is, by the definition, the stochastic solution of Eq. (18). Our aim is to find a corresponding Ito-type equation, which will be satisfied by the same process x_t. At the end we compute the difference between $g(x)dN$ and $g(x)d\circ N$. Using Eq. (14) with $H(u) = g(x(t,u))$ and $\partial/\partial u = g(x)\partial/\partial x$ we get

$$\begin{aligned} gdN - gd\circ N &= \lambda dtE\int_N^{N+z} g(x(t,u))du \\ &= \lambda dtE\int_N^{N+z} g(x(t,u))\frac{\partial u}{\partial x}dx(t,u) \\ &= \lambda dtE[x(t,N+z) - x(t,N)] \\ &= \lambda dtE\left[\exp\left(z\frac{\partial}{\partial N}\right) - 1\right]x(t,N) \\ &= \lambda dtE\int_0^z d\zeta \exp\left[\zeta g\frac{\partial}{\partial x}\right]g. \end{aligned} \qquad (19)$$

The integrand of the last expression is a solution of a partial differential equation

$$\partial\Psi/\partial\zeta = g(x)\partial\Psi/\partial x,$$

with the initial condition $\Psi(\zeta=0) = g(x)$. Thus

$$\exp\left[\zeta g\frac{\partial}{\partial x}\right]g = \Psi(\eta(x)+\zeta), \qquad (20)$$

where

$$\eta(x) = \int^x \frac{dv}{g(v)}, \quad \Psi(\eta(x)) \equiv g(x). \qquad (21)$$

Concluding, we have found that Stratonovich equation (18) is equivalent to the following Ito-type equation

$$dx_t = [f(x) + \Delta(x)]dt + g(x)d\circ N_t, \qquad (22)$$

where

$$\Delta(x) = \lambda E\int_0^z d\zeta\Psi(\eta(x)+\zeta), \qquad (23)$$

and where Ψ and η are *implicitly* given by Eqs. (21). Similarly, Ito equation

$$dx_t = f(x)dt + g(x)d\circ N_t \qquad (24)$$

corresponds to

$$dx_t = [f(x) - \Delta(x)]dt + g(x)dN_t \qquad (25)$$

in the Stratonovich interpretation. At the Wiener process limit (13) $\Delta(x)\to Dg(x)g'(x)$, achieving the form of the "spurious drift" term, well known from the ordinary Langevin equation theory.

Examples. In a number of special cases Eqs. (21) can be solved and $\Delta(x)$, Eq. (23), can be *explicitly* computed. As an example, let us find the Ito-type equation satisfied by the process $x_t = (\alpha N_t/2)^2$. One has

$$dx_t = \alpha x^{1/2} dN_t . \tag{26}$$

Thus $\eta(x) = 2x^{1/2}/\alpha = 2g(x)/\alpha^2$ and $\Psi(\eta) = \alpha^2 \eta/2$. Equation (23) takes a form

$$\Delta(x) = \lambda E \int_0^z d\zeta \frac{\alpha^2}{2} [\eta(x) + \zeta] = \lambda(\alpha x^{1/2} Ez + \alpha^2 Ez^2/4).$$

Therefore

$$dx_t = \lambda(\alpha x^{1/2} Ez + \alpha^2 Ez^2/4) dt + \alpha x^{1/2} d\circ N_t . \tag{27}$$

Calculating directly the averages $d\langle x_t \rangle = (\alpha/2)^2 d\langle N_t^2 \rangle = (\alpha/2)^2 [\lambda Ez^2 + 2(\lambda Ez)^2 t] dt = \langle \Delta(x_t) \rangle dt$ and taking into account the nonanticipating property, we verify that Ito equation (27) is correct.

In the second example we find the Ito-form corresponding to the important equation

$$dx_t = yx dN_t . \tag{28}$$

Here, $\eta(x) = y^{-1} \ln x$, $\Psi(\eta) = ye^{y\eta}$ and $\Delta(x) = \lambda x(Ee^{yz} - 1)$. We obtain

$$dx_t = \lambda x(Ee^{yz} - 1) dt + yx d\circ N_t . \tag{29}$$

Using the *explicit* solution $x_t = x_0 e^{yN_t}$ and Eq. (4) one has $d\langle x_t \rangle = \lambda \langle x_t \rangle (Ee^{yz} - 1) dt$ in agreement with Eq. (29).

A following equation

$$dS_t = rSdt + \sigma S d\circ N_t \equiv [r - \lambda(Ee^{\sigma z} - 1)] S dt + \sigma S dN_t \tag{30}$$

may be thus proposed as an extension of Bachelier-Black-Scholes equation for "Poissonian markets." The stochastic solution

$$S_t = S_0 \exp\{[r - \lambda(Ee^{\sigma z} - 1)]t\} \exp(\sigma N_t) \tag{31}$$

immediately follows from the Stratonovich equation and then can be directly applied in a financial analysis, e.g., for option pricing. Such modeling of prices seems attractive, reflecting a discrete and random time of successive transactions. Moreover, the *internal* degrees of freedom of a compound Poisson process, related to the possible choice of different $\pi(z)$, give probably the ability for a better fitting of the real data than a single parameter, volatility σ, of a Gaussian market model.

IV. FINAL REMARKS

We have shown that the Ito-type formalism can also be constructed for the Langevin equation

$$dx_t/dt = f(x) + g(x) \xi_t \tag{32}$$

with multiplicative white shot noise (WSN), $\xi_t = dN_t/dt$, where N_t is a compound Poisson process. The multiplicative noise is usually considered as the external one, introduced by the fluctuation of parameters. The definite sign of parameter is frequently required from its physical meaning or from global stability conditions [8]. In such case the use of the GWN to describe the fluctuation is not possible. Moreover the WSN appears as a well-defined (uncorrelated, and leading to the Markovian description) limit of the Campbell process. The last one is usually considered as a good description of the real physical fluctuation [11]. At the general level of the stochastic processes theory the Ito and Stratonovich approaches are equivalent in a sense that the difference may be counted by the appropriate modification of the drift term in Eq. (32). Nevertheless the specific general properties of the Ito formalism appear more useful if kinetic equation (32) is not *a priori* known, but has to be proposed in accordance with certain assumptions. For example, the linear relaxation is not affected if the multiplicative noise coupling is treated according to the Ito interpretation. Similarly, the martingale property, which is essential for the games theory, is conserved if the Ito integrals are used [1].

The most important results of the paper are Eq. (12) specifying the (Ito-type) rule of martingale integration with respect to compound Poisson process and Eqs. (23) and (21) providing the "spurious drift" term, transforming between the Ito equation and the Stratonovich equation with multiplicative white shot noise.

[1] K. Ito, Mem. Am. Math. Soc. **4**, 289 (1951); Y.V. Prokhorov and Y.A. Rozanov, *Probability Theory* (Springer, Berlin, 1969); I.I. Gihman and A.V. Skorohod, *Stochastic Differential Equations* (Springer, Berlin, 1972).
[2] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).
[3] V. Kontorovich and V. Lyandres, IEEE Trans. Signal Process. **43**, 2372 (1995).
[4] G.Q. Cai and Y.K. Lin, Phys. Rev. E **54**, 299 (1996).
[5] S. Primak and V. Lyandres, IEEE Trans. Signal Process. **46**, 1229 (1998); S. Primak, Phys. Rev. E **61**, 100 (2000).
[6] L. Bachelier, Ann. Sci. Ec. Normale Super. **17**, 21 (1900); F. Black and M. Scholes, J. Polit. Economy **81**, 637 (1973).
[7] R.N. Mantegna and H.E. Stanley, *Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, 1999).
[8] J.M. Sancho, M. San Miguel, L. Pesquera, and A.M. Rodriguez, Physica A **142**, 532 (1987); R. Zygadło, Phys. Rev. E **54**, 5964 (1996).
[9] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967).
[10] C. van den Broeck, J. Stat. Phys. **31**, 447 (1983).
[11] R. Zygadło, Phys. Rev. E **47**, 106 (1993); **47**, 4067 (1993).